Free Scalar Partition Functions

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1 Introduction

I assume that no part of this note is new to the literature, but the whole may be useful for its pedagogy and for its unified presentation of some facts that are usually scattered across the literature.

This note attempts to clarify some properties of scalar partition functions in d dimensions. Particular cases of interest are the conformally coupled scalar, the compact scalar, and the massive scalar. I will focus primarily on the backgrounds S^d and $S^{d-1} \times \mathbb{R}$, and I will use R for the radius of the respective sphere. The Ricci scalar of S^d is

$$\mathcal{R} = \frac{d(d-1)}{R^2} \tag{1}$$

The Euclidean free scalar Lagrangian with mass m and curvature coupling ξ is

$$L = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi \, \mathcal{R} \, \phi^2 \tag{2}$$

The corresponding action on a Euclidean manifold M is

$$S = \int_M d^d x \sqrt{g} \, L \tag{3}$$

The kinetic operator is

$$\Delta = -\nabla^2 + m^2 + \xi \, \mathcal{R} \tag{4}$$

In our cases of interest $\mathcal{R}$ will be non-negative, so generically nonzero m, ξ will lift the zero mode. If $m = \xi = 0$ then there is precisely one zero mode, and the partition function will diverge unless we take ϕ to be periodic. Its period is a new physical parameter of the system.

Recall that after setting $\hbar = c = 1$ the mass dimension of a boson is

$$[\phi] = \frac{d-2}{2} \tag{5}$$

Note the curvature coupling is dimensionless, $[\xi] = 0$.

2 Weyl transformation

Once we have a notion of putting the same theory on different backgrounds, the notion of a Weyl transformation is well-defined. We simply choose a reference metric $\bar{g}_{\mu\nu}$ and consider the family of metrics that can be written as

$$g_{\mu\nu} = e^{2\omega}\bar{g}_{\mu\nu} \tag{6}$$

and then we ask how a given quantity depends on ω. In the quantum theory this can be subtle.

Perhaps the simplest example is the Weyl variation of the classical action. Using

$$\begin{aligned} \delta_W \phi &= -[\phi]\delta\omega\,\phi = \frac{2-d}{2}\delta\omega\,\phi \\ \delta_W \sqrt{g} &= d\,\delta\omega\sqrt{g} \\ \delta_W g^{\mu\nu} &= -2\delta\omega\, g^{\mu\nu} \\ \delta_W \mathcal{R} &= 2(1-d)\nabla^2\delta\omega - 2\delta\omega\,\mathcal{R} \end{aligned} \tag{7}$$

one finds

$$\begin{aligned} \delta_W S &= \int_M d^d x\sqrt{g}\left(\frac{2-d}{2}\phi\nabla^\mu\phi\nabla_\mu + \frac{1}{2}m^2\phi^2 + \xi\phi^2(1-d)\nabla^2\right)\delta\omega \\ &= \int_M d^d x\sqrt{g}\left(\left[\frac{2-d}{2} - 2\xi(1-d)\right]\phi\nabla^\mu\phi\nabla_\mu + \frac{1}{2}m^2\phi^2\right)\delta\omega \end{aligned} \tag{8}$$

We see this vanishes iff $m = 0$ and

$$\xi = \frac{d-2}{4(d-1)} \tag{9}$$

This is called the conformal coupling. See e.g. (9.89) of [6]. The need for $m = 0$ is very intuitive: any nonzero m would set a physical scale, making scale-invariance hopeless.

The (negative) log partition function is the quantum analogue of the classical action, and in even dimensions it is not scale-invariant even at conformal coupling. This failure can be traced to the path integral measure, and it is called a conformal anomaly.

3 CFT

On a manifold M with metric $g_{\mu\nu}$, the (Euclidean) stress tensor is defined (up to contact terms) by[1]

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\big\langle \dots \big\rangle_M = \frac{1}{2}\big\langle T^{\mu\nu}(x)\dots\big\rangle_M \tag{10}$$

where $\langle\dots\rangle_M$ is a correlator of local operators computed in the background M. In the special case of an infinitesimal Weyl transformation, $\delta g_{\mu\nu} = 2\delta\omega\, g_{\mu\nu}$, we have

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta\omega(x)}\log\big\langle \dots \big\rangle_M = \frac{\big\langle T^\mu{}_\mu(x)\dots\big\rangle_M}{\big\langle \dots \big\rangle_M} \tag{11}$$

CFT's are characterized by classical Weyl-invariance, meaning that $T^\mu{}_\mu$ vanishes up to anomaly. It turns out that the anomalous operator $T^\mu{}_\mu(x)$ is a c-number, decoupling from all other local operators. (I.e. locally it's a multiple of the identity.) Therefore the other insertions on the right-hand of (11) cancel out, and we're

[1] This is the field theory convention. The string theory convention differs by a factor of -2π. See (1.2.22), (3.4.4) of [13] and (3.189) of [4]. Keep in mind also that the Lorentzian and Euclidean cases differ by a sign.

left with the c-number $T^\mu{}_\mu$. Rewriting slightly and specializing to the partition function $Z(M) \equiv \langle 1 \rangle_M$, we have

$$\delta_W \log Z(M) = \int d^d x \sqrt{g}\, \delta\omega(x)\, T^\mu{}_\mu(x) \tag{12}$$

There is extensive literature on how to compute the trace anomaly $T^\mu{}_\mu(x)$, also known as the Weyl anomaly. In general it consists of curvature invariants. In $d = 2$ we have simply [13][2]

$$d = 2: \qquad T^\mu{}_\mu = \frac{c}{24\pi}\mathcal{R} \tag{13}$$

where c is the central charge. In odd dimensions the anomaly vanishes. In $d = 4$ we have (see e.g. (6.112) and (6.105) of [4])

$$d = 4: \qquad T^\mu{}_\mu = a_4 E_4 + \frac{b_4}{32\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{d_4}{32\pi^2} \nabla^2 \mathcal{R} \tag{14}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and E_4 is the 4D Euler density[3]

$$E_4 = \frac{1}{32\pi^2}\left(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2\right) \tag{15}$$

My convention for the coefficients a_4, b_4, d_4 is nonstandard. Their values depend on the particular CFT in question. Using e.g. (6.112) and (6.106) of [4], for the conformally coupled scalar they are

$$a_4 = -\frac{1}{180}, \qquad b_4 = \frac{1}{60}, \qquad d_4 = \frac{1}{90} \tag{16}$$

The general structure of the trace anomaly is described on pg. 10 of [7]. (See [11] for a more modern discussion.) There is one type A term, proportional to the Euler density E_d. The B terms correspond to all the Weyl invariants one can construct from the Weyl tensor and its derivatives. Their number increases with dimension. In $d = 2$ there are no type B terms, in $d = 4$ there is one, and in $d = 6$ there are three. There can also be total derivative terms, referred to as type D anomalies in [11], which can be removed by conformally non-invariant local counterterms. Most authors choose to remove them, but we have kept them for no good reason.[4] For the general case let's write

$$T^\mu{}_\mu = a_d E_d + \sum_i b_{d,i} I_i + \sum_j d_{d,j} J_j \tag{17}$$

where the I_i are constructed from the Weyl tensor and its derivatives. I don't know much about the type D terms. Recalling that the $(1,3)$-valent Weyl tensor $C^\mu{}_{\nu\rho\sigma}$ is Weyl-invariant, we can see that the $d = 4$ B term $\sqrt{g}\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ is indeed Weyl-invariant.

In the special case of constant $\delta\omega$, i.e. a dilation, the type D terms drop out and on a Weyl-flat manifold we have

$$\begin{aligned} \frac{\delta}{\delta\omega} \log Z(M) &= \int d^d x \sqrt{g}\, T^\mu{}_\mu \\ &= a_d\, \chi(M) \end{aligned} \tag{18}$$

where $\chi(M)$ is the Euler characteristic. Consider the case of a conformal scalar on a sphere of radius R as a check. In $d = 2$ the Euler density is $E_2 = \frac{1}{4\pi}\mathcal{R}$ so apparently $a_2 = \frac{1}{6}c$. Recall also that $\chi(S^2) = 2$. Then

[2] It's actually possible to also have a constant term in the trace, coming from a cosmological constant as in (3.4.25) of [13]. This is not Weyl-invariant, but it's state-independent and really just a number, so we still consider the CFT a CFT. People almost always use a scheme where it's set to zero.

[3] The Euler density is a curvature invariant whose integral (in my convention) gives the Euler characteristic, and in $d = 2n$ dimensions it involves n powers of the Riemann tensor. See [12] for more discussion.

[4] Different regularization schemes (might) give different values of d_4, which means we're *forced* to introduce the counterterm $\int R^2$. This breaks Weyl invariance, but I guess it's state-independent so maybe its effect is mild enough that we'd still consider the theory a CFT. Similar to the footnote above for trace anomaly in 2D CFT. See also 2404.15561 for 6D total derivative effects in the Casimir energy.

$\frac{\delta}{\delta\omega}\log Z(S^2) = \frac{1}{3}$ and consequently $\log Z(S^2) \sim \frac{1}{3}\log R$, which is consistent with the familiar 2D CFT result that $\log Z(S^2) \sim \frac{c}{3}\log R$.

On a product space $M \times N$ we have $\chi(M \times N) = \chi(M) \cdot \chi(N)$, and we know $\chi(S^1) = 0$, so for any M we have $\chi(M \times S^1) = 0$ and therefore for any Weyl-flat M the thermal partition function $Z(M \times S^1)$ is invariant under dilations. In particular, all round spheres are Weyl-flat so $Z(S^{d-1} \times S^1)$ is invariant under dilations.

The anomaly completely determines the trace of the stress tensor, but it also affects the other components. The stress tensor vanishes in flat space $\mathbb{R}^d$, so in Weyl-related spaces the whole stress vev is determined by the type A and D anomalies. The results for $d = 2, 4$ can be found in [5]. The cylinder case $M = S^{d-1} \times S^1$ is of particular interest, and the type A contribution for general d can be found in [11]. As a corollary they compute the Casimir energies. The $d = 2$ result is

$$E_{\text{Casimir}}(S^1) = -\frac{c}{12R} = -\frac{a_2}{2R} \tag{19}$$

and the $d = 4$ result for the conformal scalar is

$$E_{\text{Casimir}}(S^3) = -\frac{3a_4}{8R} + \text{type D} = \frac{1}{240R} \tag{20}$$

4 Partition function on S^d

From above, our kinetic operator is

$$\Delta = -\nabla^2 + m^2 + \frac{d(d-1)}{R^2}\xi \tag{21}$$

Case $d = 2$

For simplicity let's restrict to special cases. First work in $d = 2$ with conformal coupling $\xi = 0$, which happens to also be minimal. We will work through it in some detail to illustrate general principles, in particular how to handle zero modes. Take ϕ to be periodic, $\phi \sim \phi + C$. Note $[C] = 0$. Work in the limit of large C so that winding modes are exponentially suppressed and we can ignore them. In this way C comes in rather trivially and is effectively an IR regulator for the would-be zero mode divergence. We expand the field in orthonormal eigenmodes of the Laplacian,

$$\phi(x) = \sum_n \phi_n f_n(x) \tag{22}$$

where

$$\int_{S^2} d^2x\sqrt{g}\, f_m(x) f_n(x) = \delta_{mn} \tag{23}$$

This means that $f_n(x)$ depends on R. The dimensions are $[f_n] = 1$ and $[\phi_n] = -1$. Note that for $n \neq 0$ the ϕ_n are real-valued, but the zero mode has $\phi_0 \sim \phi_0 + \sqrt{4\pi R^2}\, C$. The action is

$$S[\phi] = \frac{1}{2}\sum_n \lambda_n \phi_n^2 \tag{24}$$

where λ_n is the eigenvalue of $f_n(x)$. We adopt the measure

$$\mathcal{D}\phi = \prod_n \mu\, d\phi_n \tag{25}$$

where μ is an arbitrary mass scale inserted to make the measure dimensionless. It's like a UV cutoff. The partition function is

$$\begin{aligned} Z(R,C,\mu) &= \int \mathcal{D}\phi\, e^{-S[\phi]} \\ &= \sqrt{4\pi}\,\mu R\, C \left(\prod_{n\neq 0} \mu^{-2}\lambda_n\right)^{-1/2} \end{aligned} \tag{26}$$

This product is divergent and must be regularized somehow. Zeta regularization (see e.g. [17]) gives

$$Z(R,C,\mu) = \sqrt{4\pi}\,\mu R\, C \times (\mu R)^{-2/3}\sigma \tag{27}$$

where σ is a computable number independent of R, C, and μ. It, and the factor of $\sqrt{4\pi}$, can be absorbed into a redefinition of μ. This leaves us with

$$\log Z(R,C,\mu) = \frac{1}{3}\log \mu R\, C^3 + \text{ non-universal} \tag{28}$$

For any 2D CFT the coefficient of $\log R$ will be $c/3$, and for the scalar we have $c = 1$ so this checks out. Zeta regularization automatically gets rid of power terms of the UV regulator μ, but the log term remains. At least in this partition function context, the explicit dependence on μ in the final answer indicates an anomaly; the supposedly conformally invariant partition function in fact depends on R. The path integral recipe here, where we expand in orthonormal eigenmodes and introduce an arbitrary scale to make the measure dimensionless, will be used repeatedly. Note finally that if we had instead used a lattice cutoff with spacing ε then the log divergence would have been $\frac{1}{3}\log\frac{R}{\varepsilon}$. If we parametrize the cutoff with the number of sites $N \sim R/\varepsilon$ rather than the spacing ε then the log divergence reads $\frac{1}{3}\log N$. This has no R dependence and so one might naïvely conclude that the coefficient of $\log R$ is zero rather than $\frac{1}{3}$. This mistake shows that one must be careful when asking about the "universal log coefficient". See section 3 of [8] for further discussion.

In $d = 2$ the case of general nonzero m, ξ is actually tractable. There are no zero modes, so we just have a determinant to compute,

$$\begin{aligned} Z(R,m,\xi,\mu) &= \det(\mu^{-2}\Delta)^{-1/2} \\ &= \left(\prod_n \mu^{-2}\lambda_n\right)^{-1/2} \\ &= \left(\prod_{\ell=0}^{\infty}\left[\frac{\ell(\ell+1)+(mR)^2+2\xi}{(\mu R)^2}\right]^{2\ell+1}\right)^{-1/2} \end{aligned} \tag{29}$$

We evaluate by taking derivatives of the log until the series converges. Note only the combinations

$$\rho \equiv (mR)^2 + 2\xi \tag{30}$$

and μR show up, so the final answer must also be a function of only these combinations. We proceed as

$$\begin{aligned} \partial_\rho^2 \log Z(\rho,\mu R) &= -\frac{1}{2}\partial_\rho^2 \sum_{\ell=0}^{\infty}(2\ell+1)\log\frac{\ell(\ell+1)+\rho}{(\mu R)^2} \\ &= -\frac{1}{2}\partial_\rho \sum_{\ell=0}^{\infty}\frac{(2\ell+1)}{\ell(\ell+1)+\rho} \\ &= \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{(2\ell+1)}{[\ell(\ell+1)+\rho]^2} \\ &= \frac{1}{2\sqrt{1-4\rho}}\Big[\psi'(\frac{1-\sqrt{1-4\rho}}{2}) - \psi'(\frac{1+\sqrt{1-4\rho}}{2})\Big] \end{aligned} \tag{31}$$

where $\psi(z) = \partial_z \log\Gamma(z)$ is the digamma function. This implies

$$\log Z(\rho, \mu R) = f(\rho) + \rho\, C_1(\mu R) + C_2(\mu R) \tag{32}$$

where C_1, C_2 are constants of integration and

$$f(\rho) = \frac{1}{2}\sqrt{1-4\rho}\left[\log\Gamma\big(\frac{1-\sqrt{1-4\rho}}{2}\big) - \log\Gamma\big(\frac{1+\sqrt{1-4\rho}}{2}\big)\right] + \psi^{(-2)}\big(\frac{1-\sqrt{1-4\rho}}{2}\big) + \psi^{(-2)}\big(\frac{1+\sqrt{1-4\rho}}{2}\big) \tag{33}$$

with $\psi^{(-2)}(z)$ the second antiderivative of $\psi(z)$, or equivalently the first antiderivative of $\log\Gamma(z)$. It's in Mathematica as PolyGamma[-2, z]. The main point is that Z primarily depends on the dimensionless combination ρ, and the only possible μ dependence comes from C_1, C_2. We can actually deduce C_1, C_2 using heat kernel methods. The heat kernel coefficient that controls UV divergences in $d = 2$ is $a_2(x)$. From (4.27) of [16], when $m = 0$ it is

$$a_2(x) = \frac{1}{24\pi}(1-6\xi)\frac{2}{R^2} = \frac{1-6\xi}{12\pi R^2} \tag{34}$$

The log UV divergent behavior of the partition function, i.e. the μ dependence, is (using μ as an actual UV cutoff now, and focusing only on small t since $a_2(x)$ is defined by the small t expansion)

$$\begin{aligned}\log Z(R,\xi,\mu) &\supset \frac{1}{2}\int_{\mu^{-2}}\frac{dt}{t}\int d^2x\sqrt{g}\,\frac{1-6\xi}{12\pi R^2} \\ &= \frac{1-6\xi}{6}\int_{\mu^{-2}}\frac{dt}{t} \\ &= \frac{1-6\xi}{3}\log\mu \;+\; \text{non-}\mu\end{aligned} \tag{35}$$

A full calculation using the exact heat kernel, as opposed to just its small-t expansion, would give a log with dimensionless argument. The only other scale around is R, so we must actually have

$$\log Z(R,\xi,\mu) \supset \frac{1-6\xi}{3}\log\mu R \tag{36}$$

Comparing with the above expression for $Z(\rho, \mu R)$ tells us

$$C_1(\mu R) = -\log\mu R, \qquad C_2(\mu R) = \frac{1}{3}\log\mu R \tag{37}$$

The full answer is then

$$\boxed{\log Z(R,m,\xi,\mu) = f(m^2R^2+2\xi) - \big(m^2R^2+2\xi-\frac{1}{3}\big)\log\mu R} \tag{38}$$

Compare with (C.21) of [3]. One important takeaway is that even without the μ terms, the partition function is not scale invariant due to m. The presence of an explicit scale in the couplings always breaks scale invariance. However rescaling all parameters, including the UV regulator, according to their dimensions leaves the partition function invariant. Note the $\frac{1}{3}\log\mu R$ part of the anomaly is the same as in the conformally coupled case.

As a nice consistency check let's recover the conformal scalar by setting $\xi = 0$ and sending $m \to 0$. Recall that for the conformal scalar the IR regulator was its dimensionless period C. Here the effect of the mass kicks in when the action term $\int d^2x \frac{1}{2}m^2\phi^2 \sim (mR)^2\phi^2$ is of order unity, so ϕ's fluctuations will be cut off around $\phi \sim \frac{1}{mR}$, and we should identify $C \sim \frac{1}{mR}$. Note that the C_1 term vanishes in the $m \to 0$ limit, and that

$$f(\rho) = -\frac{1}{2}\log(1-\sqrt{1-4\rho}) + \mathcal{O}\big((1-\sqrt{1-4\rho})^0\big) = -\log mR + \mathcal{O}\big((mR)^0\big) \tag{39}$$

We are left with

$$\log Z(R,m,\mu) = \log\frac{1}{mR} + \frac{1}{3}\log\mu R + \mathcal{O}\big((mR)^0\big) \tag{40}$$

Identifying $\frac{1}{mR} \sim C$ we see that this agrees with the 2D conformal scalar calculation above in (28).

Case $d > 2$

Next move onto $d > 2$, where we will be less explicit. The conformally coupled scalar in odd dimensions has only the parameters R, μ. Its partition function has no log term, and accordingly the constant piece (independent of R, μ) is universal. The lack of R-dependence is as it should be for the partition function of a non-anomalous conformally invariant theory. Note that since μ only shows up in the measure $\prod_n \mu\, d\phi_n$, and μ drops out, the measure itself must be scale-invariant.

The periodic scalar (necessarily massless and minimally coupled) in odd d has only the parameters R, C, μ. Recall $[C] = [\phi] = \frac{d-2}{2}$ and the mode coefficients have $[\phi_n] = -1$. The partition function has a zero mode part and a functional determinant part,

$$Z(R,C,\mu) = \mu C R^{d/2} \times \det{}'(-\mu^{-2}\nabla^2)^{-1/2} \tag{41}$$

The prime on the determinant indicates that we omit zero modes. According to (16) of [9], in odd dimensions pulling a factor out of the Laplacian determinant results in its reciprocal factor outside. This implies

$$\begin{aligned} Z(R,C,\mu) &= \mu C R^{d/2} \times \frac{1}{\mu R}\det{}'(-R^2\nabla^2)^{-1/2} \\ &= C R^{\frac{d-2}{2}}\det{}'(-R^2\nabla^2)^{-1/2} \end{aligned} \tag{42}$$

Now μ has cancelled out entirely, and note that $R^2\nabla^2$ is independent of R. The partition function still has R-dependence though, stemming from the zero mode contributing a different factor of R than the rest of the modes. It's also intuitive that the dimensionful scale $[C]$ broke scale invariance [1].

Consider the conformal scalar in $d = 4$ and recall $\chi(S^4) = 2$. The partition function depends only on R, μ. We know from above that its scaling is set by $a_4\chi(S^4) = -1/90$, i.e.

$$\log Z(R,\mu) = -\frac{1}{90}\log\mu R + \text{non-universal} \tag{43}$$

For the conformal scalar partition function in general even d the log coefficient $2a_d$ is the only physical observable.

Some lessons are as follows. Only dimensionless combinations of the physical parameters (including the UV regulator) can show up. The UV regulator μ only shows up through anomaly, and only in very simple ways such as a log. When R is the only parameter besides μ (as in the CFT case), the partition function really does have only one number's worth of information. The PI measure is scale-invariant up to anomaly. There is still anomaly with a mass present, although there is also richer dependence on mR. Under dilations the field does not inherently transform according to its dimension, but this is not a problem thanks to the scale-invariance of the measure (up to anomaly), and the classical action is what naturally picks out the dimension of the operator.

5 Partition function on $S^{d-1} \times S^1$

For simplicity I'll not consider twists around the S^1 (in the $d = 2$ case this means I only consider rectangular tori). Let the proper length of the S^1 be β. Once again let's split into cases, starting with the 2D CFT case with large period $\phi \sim \phi + C$. From (7.2.7-10) of [13] we have

$$Z(\beta, R, C, \mu) = \frac{C}{\sqrt{\beta/R}} |\eta(\frac{i\beta}{2\pi R})|^{-2} \tag{44}$$

where the Dedekind eta function is

$$\eta(i\tau_2) = (e^{-2\pi\tau_2})^{1/24} \prod_{n=1}^{\infty} (1 - e^{-2\pi n\tau_2}) \tag{45}$$

Note β, R only show up in the combination β/R, and there is no UV regulator in the final answer. This is not a coincidence. From the CFT section above we know that rescaling β, R by the same factor, i.e. dilating the full manifold, while keeping other parameters fixed must leave the partition function invariant because $\chi(T^2) = 0$. But we also know that rescaling all parameters according to their respective dimensions, including the UV regulator μ, always leaves the partition function invariant. Together these facts imply that the torus partition function is independent of μ. A similar argument applies for general d.

Another thing to note about this partition function is its large β behavior,

$$Z \sim \frac{1}{\sqrt{\beta}} \exp\left[\frac{\beta}{12R}\right] \tag{46}$$

We expect the coefficient of $-\beta$ in the exponent to be the vacuum energy, and indeed it agrees with the Casimir energy (19).

6 Flat space

Despite being the most familiar and well-studied case, flat space is in many ways pathological. Most authors are very cavalier about its subtleties. For example consider Polyakov's famous non-local effective action for the anomaly in 2D CFT [14], reviewed for example in (3.4.19) of [13]. It states that for a conformally flat metric g one has

$$Z[g_{\mu\nu}] = Z[\delta_{\mu\nu}] \exp\left(\frac{a_1}{8\pi} \int d^2x \int d^2x' \sqrt{g(x)} R(x) G(x, x') \sqrt{g(x')} R(x')\right) \tag{47}$$

where $G(x, x')$ is the Green's function for the Laplacian with respect to g. Under a rigid dilation both $\sqrt{g}R$ and the Green's function are invariant, so this formula suggests that the partition function must be invariant as well. In the case of a sphere this is clearly in contrast with the discussion above. But many authors like to claim that the plane is really just a big regulated sphere, so something is fishy. The resolution is related to the fact that the Laplacian is not invertible on closed manifolds, whereas for an asymptotically flat space we discard the constant mode as being non-normalizable. This is discussed in detail in [10].

Related to this, one may wonder how dilation-invariance on the plane is consistent with dilation non-invariance on closed manifolds. It turns out that one can indeed view the plane as a big regulated sphere, but the dilation of the plane gets regulated into a certain global conformal transformation on the sphere, rather than a dilation of the sphere.

See e.g. [2] for careful calculations of partition functions in flat space.

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